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THE ATTRACTION BETWEEN COILS IN THE RAYLEIGH CURRENT BALANCE

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ABSTRACT

A formula is derived for the force between two parallel coaxial coils whose winding channels have rectangular cross sections. The expression for the maximum force also is obtained. The linear dimensions of the cross section of the windings are all small compared with the mean radii of the coils and sixth-order terms in these ratios are neglected. The principal term in the force corresponds to two linear circular currents in the mean positions of the coils. The remaining terms take account of (*a*) the finite cross section of the winding channels, (*b*) the discrete nature of the windings (round insulated wires) with waste space between them, and (*c*) the nonuniformity of current over the section of the wire. The effects (*b*) and (*c*) are eliminated when the ratio of the mean radii of the coils is determined electrically.

CONTENTS

I. INTRODUCTION

The absolute measurement of an electric current by the use of a certain type of current balance requires the computation of the force $\mathfrak{F}$ between two coaxial coils in which electric currents I_1 and I_2 are circulating, and when the coils are so spaced as to make this force a maximum, which is then indicated by $\mathfrak{F}_m$.

If the distance between their mean planes is in the general case z (and z_m when the force is a maximum), and if their mean radii are a_1 and a_2, and if the axial breadth $2b_i$ and radial depth $2c_i$ (for $i=1,2$) of the rectangular cross sections of the winding channel are such that the quantities b_i and c_i are small compared to either of the three finite quantities a_1, a_2, or z, then the principal part of this force is n_1, n_2, I_1 I_2 F where n_i is the number of turns and F is the force between two circular, coaxial, unit-current filaments having the mean radii and lying in the mean planes z_1 and z_2 of the coils. This force F is a function of two dimensionless variables x and α.

$$\alpha\equiv\frac{a_2}{a_1}<1,\ x\equiv\frac{z}{A},\ A^2\equiv a_1^2+a_2^2 \tag{1}$$

where $z=z_1-z_2$. When x and α are known, $F(x, \alpha)$ may be computed with great precision by Maxwell's formula in terms of the complete elliptic integrals K and E with modulus k where

$$k^2=\frac{4a_1a_2}{(a_1+a_2)^2+z^2}=\frac{2\sqrt{1-\beta^2}}{1+x^2+\sqrt{1-\beta^2}} \text{ and } \beta\equiv\frac{1-\alpha^2}{1+\alpha^2} \tag{2}$$

$$F(x,\alpha)=\frac{\pi zk}{\sqrt{a_1a_2}}\left[\frac{2-k^2}{1-k^2}E-2K\right]=2\sqrt{2}\pi^2x(1-\beta^2)^{-\frac{1}{4}}k^4\phi'(k^2) \tag{3}$$

where $\phi'(k^2)$ denotes $d\phi/dk^2$ and

$$\phi=\frac{2}{\pi k}(K-E)-\frac{kK}{\pi} \tag{4}$$

The function ϕ satisfies the differential equation

$$k^4(1-k^2)\phi''=k^4\phi'+\frac{3}{4}\phi \tag{5}$$

The force $\mathfrak{F}$ between the actual coils obtained by integration over their cross sections will be expressible in the form

$$\mathfrak{F}(x,\alpha)=n_1n_2I_1I_2F(x,\alpha)[1+\Delta_2(x,\alpha)+\Delta_4(x,\alpha)\ .\ .\ .] \tag{6}$$

where the functions of x and α designated by Δ_2 and Δ_4 are correction terms of the second and fourth order in the small constant ratios b_1/a_1, c_1/a_1, etc., and are to be found.

II. THE FORCE WHEN THE CURRENT UNIFORMLY FILLS THE WINDING SPACES

The force between two current filaments of radii r_1 and r_2 planes z_1' and z_2' may be expanded by Taylor's theorem in terms of the small quantities $z_1'-z_1$, $z_2'-z_2$, r_1-a_1, r_2-a_2. Integrating this over the two cross sections gives on neglecting terms of higher order than the fourth,

$$\begin{aligned}\frac{\mathfrak{F}}{n_1n_2I_1I_2}=F&+\frac{1}{6}\left\{(b_1^2+b_2^2)D_z^2F+c_1^2D_{a_1}^2F+c_2^2D_{a_1}^2F\right\}\\&+\frac{1}{360}\Big\{[3(b_1^4+b_2^4)+10b_1^2b_2^2]D_z^4F+3(c_1^4D_{a_1}^4F+c_2^4D_{a_2}^4F) \qquad (7)\\&+10c_1^2c_2^2D_{a_1}^2D_{a_2}^2F+10(b_1^2+b_2^2)(c_1^2D_{a_1}^2D_z^2F+c_2^2D_{a_2}^2D_z^2F)\Big\}\end{aligned}$$

where the partial derivatives of F imply it is a function of three variables z, a_1, and a_2.

A first step in reducing this to a workable formula is to express all the partial derivatives in terms of z-derivatives and then these in terms of x-derivatives. Let F^{I}, F^{II}, etc., denote D_xF, D_x^2F, etc., so

that $D_z{}^nF=F^{(n)}/A^n$. Then it is found by the use of equations (1) to (5) that

$$\left.\begin{aligned} D_{a_1}{}^2F&=-\frac{F^{\text{II}}}{A^2}-\frac{1}{2a_1^2}\left[\left(x-\frac{\beta}{x}\right)F^{\text{I}}+\frac{\beta}{x^2}F\right]\\ D_{a_2}{}^2F&=-\frac{F^{\text{II}}}{A^2}-\frac{1}{2a_2^2}\left[\left(x+\frac{\beta}{x}\right)F^{\text{I}}-\frac{\beta}{x^2}F\right]\end{aligned}\right\}\qquad(8)$$

By use of these, all the terms of second order in (7) may be expressed in terms of F, F^{I}, and F^{II}.

For the partial derivatives of fourth order which appear in (7), one finds.

$$\begin{aligned} D_{a1}^2D_z{}^2F&=-\frac{F^{\text{IV}}}{A^4}-\frac{1}{2A^2a_1^2}\left[\left(x-\frac{\beta}{x}\right)F^{\text{III}}+\left(2+\frac{3\beta}{x^2}\right)F^{\text{II}}-\frac{6\beta}{x^3}F^{\text{I}}+\frac{6\beta}{x^4}F\right]\\ D_{a2}^2D_z{}^2F&=-\frac{F^{\text{IV}}}{A^4}-\frac{1}{2A^2a_2^2}\left[\left(x+\frac{\beta}{x}\right)F^{\text{III}}+\left(2-\frac{3\beta}{x^2}\right)F^{\text{II}}+\frac{6\beta}{x^3}F^{\text{I}}-\frac{6\beta}{x^4}F\right]\end{aligned}\qquad(9)$$

It is also found that

$$\left.\begin{aligned} D_{a_1}^4F&=-\frac{F^{\text{IV}}}{A^4}+\frac{F^{\text{II}}}{A^2a_1^2}-2D_{a_1}^2D_z^2F\\ D_{a_2}^4F&=-\frac{F^{\text{IV}}}{A^4}+\frac{F^{\text{II}}}{A^2a_2^2}-2D_{a_2}^2D_2^2F\end{aligned}\right\}\qquad(10)$$

and

$$D_{a_1}^2D_{a_2}^2F=-\frac{F^{\text{IV}}}{A^4}-D_{a_1}^2D_z^2F-D_{a_2}^2D_z^2E+\frac{1}{2a^2a^2}[(1+x^2)F^{\text{II}}+3xF^{\text{I}}]\qquad(11)$$

By use of (9) to (11) the term of fourth order in (7) is expressible in terms of F, F^{I}, $F^{\text{II}}\ldots F^{\text{IV}}$. It is then possible to express F^{II}, F^{III}, and F^{IV} in terms of F and F^{I}, for it is found from (1) to (5) that if a_1 and a_2 are constant the force F satisfies the ordinary differential equation in x.

$$(x^4+2x^2+\beta^2)x^2F^{\text{II}}+(5x^4+2x^2-3\beta^2)xF^{\text{I}}-(2x^2-3\beta^2)F=0\qquad(11)'$$

From this and the equation found by differentiating it successively, all derivatives of higher order than the first may be expressed in terms of F and F^{I}. However, for the purpose of computation, it is simpler not to make this elimination but to compute in succession the functions of x and α, λ_1, $\lambda_2\ldots\lambda_4$, which are defined by

$$\left.\begin{aligned} \lambda_1&\equiv\frac{xF^{\text{I}}}{F}\\ \lambda^2&\equiv-\frac{x^2F^{\text{II}}}{F}=\frac{3\beta^2-2x^2+(5x^4+2x^2-3\beta^2)\lambda_1}{x^4+2x^2+\beta^2}\\ \lambda_3&\equiv\frac{x^3F^{\text{III}}}{F}=\frac{4x^2-(25x^4+4x^2)\lambda_1+(11x^4+10x^2-\beta^2)\lambda_2}{x^4+2x^2+\beta^2}\\ \lambda_4&\equiv-\frac{x^4F^{\text{IV}}}{F}=\frac{x^2[75x^2\lambda_1-3(23x^2+8)\lambda_2+16(x^2+1)\lambda_3]}{x^4+2x^2+\beta^2}\end{aligned}\right\}\qquad(12)$$

In this way the expression (7) leads to the form (6) where

$$\Delta_2(x,\ \alpha)=\frac{1}{12x^2}\left\{-\left[\frac{c_1^2}{a_1^2}-\frac{c_2^2}{a_2^2}\right]\beta-\left[(x^2-\beta)\ \frac{c_1^2}{a_1^2}+(x^2+\beta)\ \frac{c_2^2}{a_2^2}\right]\lambda_1 - \frac{2[b_1^2-c_1^2+b_2^2-c_2^2]}{A^2}\ \lambda_2\right\} \quad (13)$$

$$\Delta_4(x,\ \alpha)=\frac{1}{360A^2x^4}\left\{B_4\lambda_4-(B_2x^2-\beta B_1)\lambda_3-\left[5\left(\frac{c_1c_2A}{a_1a_2}\right)^2x^4+B_3x^2-3\beta B_1\right]\lambda_2 +3\left[5\left(\frac{c_1c_2A}{a_1a_2}\right)^2x^4+2\beta B_1\right]\lambda_1-6\beta B_1\right\} \quad (14)$$

where

$$\left.\begin{aligned}
B_1&=\frac{c_1^2}{a_1^2}\ [5(b_1^2+b_2^2-c_2^2)-3c_1^2]-\frac{c_2^2}{a_2^2}\ [5(b_1^2+b_2^2-c_1^2)-3c_2^2]\\
B_2&=\frac{c_1^2}{a_1^2}\ [5(b_1^2+b_2^2-c_2^2)-3c_1^2]+\frac{c_2^2}{a_2^2}\ [5b_1^2+b_2^2-c_1^2)-3c_2^2]\\
B_3&=\frac{c_1^2}{a_1^2}\ [9c_1^2+15c_2^2-10(b_1^2+b_2^2)]+\frac{c_2^2}{a_2^2}\ [9c_2^2+15c_1^2-10(b_1^2+b_2^2)]\\
B_4&=\frac{1}{A^2}\ [-3(b_1^4+c_1^4+b_2^4+c_2^4)+10(b_1^2c_1^2+b_2^2c_2^2)-10(b_1^2-c_1^2)(b_2^2-c_2^2)]
\end{aligned}\right\} \quad (15)$$

III. THE MAXIMUM FORCE

In the absolute measurement of current, the distance between the mean planes of the coils is adjusted by trial to a value z_m which makes $\mathfrak{F}$ a maximum. The corresponding value of x is x_m. The value z_o (and x_o corresponding) which makes $\mathfrak{F}$ a maximum is slightly different on account of the correction terms Δ_2 and Δ_4. If primes denote x-derivatives as before, the equation to determine x_m is

$$\frac{\mathfrak{F}^1(x_m,\ \alpha)}{\mathfrak{F}(x_m,\ \alpha)}=\frac{\mathfrak{F}^1(x_m,\ \alpha)}{\mathfrak{F}(x_m,\ \alpha)}+\frac{\Delta_2{}^1(x_m,\ \alpha)+\Delta_4{}^1(x_m,\ \alpha)}{1+\Delta_2(x_m,\ \alpha)}=0 \quad (16)$$

whereas the corresponding equation determining x_o is

$$\frac{F^1(x_0,\alpha)}{F(x_0\alpha)}=0 \quad (16)'$$

Writing $x_m=x_o+\delta x_o$ in (16) and expanding the functions on the right in ascending powers of δx_o, and taking account of (16)′, shows that δx_o is a second-order quantity whose principal part is

$$\delta x_0=\frac{x_0^2\Delta'_2(x_0,\alpha)}{\lambda_2\ (x_0,\alpha)} \quad (17)$$

Consequently the maximum force $\mathfrak{F}_m=\mathfrak{F}(x_m,\alpha)$ is given to the fourth order inclusive by

$$\mathfrak{F}_m=n_1n_2I_1I_2F(x_0,\alpha)\left[1+\Delta_2(x_0,\alpha)+\Delta_4(x_0,\alpha)+\frac{(x_0\Delta'_2(x_0,\alpha))^2}{2\lambda_2(x_0,\alpha)}\right] \quad (18)$$

where

$$x_0\Delta_2{}'(x_0,\alpha)=\frac{1}{12x_0{}^2}\left\{2\beta\left(\frac{c_1^2}{a_1^2}-\frac{c_2^2}{a_2^2}\right)+\left[(x_0^2-\beta)\ \frac{c_1^2}{a_1^2}+(x_0^2+\beta)\ \frac{c_2^2}{a_2^2}\right]\lambda_2(x_0,\alpha)\right.$$

$$+2\frac{(b_1^2-c_1^2+b_2^2-c_2^2)}{A^2}\lambda_3(x_0,\alpha)\Bigg\} \tag{19}$$

$$\Delta_2(x_0,\alpha)=\frac{1}{12x_0^2}\left\{-\left(\frac{c_1^2}{a_1^2}-\frac{c_2^2}{a_2^2}\right)\beta-2\frac{(b_1^2-c_1^2+b_2^2-c_2^2)}{A^2}\lambda_2(x_0,\alpha)\right\} \tag{20}$$

The term $\Delta_4(x_o,\alpha)$ is computed from (14) and (15) for the value $x=x_o$ which makes $\lambda_1=0$ so that by (12) the functions λ_2, λ_3, and λ_4 become rational fractions in $x_o{}^2$ and β^2.

IV. THE FRACTIONAL ERROR IN THE MAXIMUM FORCE

In evaluating the small terms Δ_2, etc., the axial width $2b_i$ and radial depth $2c_i$ of the winding section must be measured. The mean radii of the coils (a_1 and a_2) may also be obtained from measurements with sufficient precision for this part of the computation. For the principal term $F(x_o)$ these enter only through their ratio $\alpha=a_2/a_1$ and this must be known with the highest precision although x_o need be known only with moderate exactness since its error makes an uncertainty of second order in the force. The fractional error in $\mathfrak{F}_m$ due to this source is

$$\left(\frac{\delta\mathfrak{F}_m}{\mathfrak{F}}\right)_{xo}=-\frac{3}{2}\cdot\frac{\beta^2-\frac{2}{3}x_0^2}{\beta^2+2x_0^2+x_0^4}\left(\frac{\delta x_o}{x_o}\right)^2\approx-.694\left(\frac{\delta x_o}{x_o}\right)^2 \text{ if } \alpha=.5 \tag{21}$$

approximately so that an error of 0.1 percent in x_o would only make an error of 1.5 parts in a million in the force $\mathfrak{F}_m$. It is evident, therefore, that when α is known, x_o may be obtained with ample precision by interpolation in the table 4 given by Dr. Grover.[1] This table gives $y_m\equiv z_o/a_1$ so that by (1)

$$x_o=\frac{y_m}{\sqrt{1+\alpha^2}} \tag{22}$$

An estimate of the fractional error introduced into the force (equation (18)) by all errors in measurement of the dimensions of the coils is found by taking the first-order variation of this equation. In doing this the δx_o is omitted since by (21) it produces no variation of first order. Also the terms of fourth order in (18) may obviously be unvaried (since they are almost negligible themselves). Furthermore, in the term Δ_2 the errors δb_i and δc_i in the measurement of the cross-sectional dimensions of the coils will be dominant, while those caused by uncertainty in a_1, a_2, and x_o will be here of no practical importance.

With this understanding, the first-order variation in $\mathfrak{F}_m$ is

$$\frac{\delta\mathfrak{F}_m}{\mathfrak{F}_m}=\frac{1}{2x_0^2}\left\{\beta\frac{\delta\alpha}{\alpha}+\frac{1}{3}\frac{c_1^2}{a_1^2}[(1+\beta)\lambda_2-\beta]\frac{\delta c_1}{c_1}+\frac{1}{3}\frac{c_2^2}{a_2^2}[(1-\beta)\lambda_2+\beta]\frac{\delta c_2}{c_2}\right.$$
$$\left.-\frac{(1+\beta)\lambda_2}{3}\frac{b_1^2}{a_1^2}\frac{\delta b_1}{b_1}-\frac{(1-\beta)\lambda_2}{3}\frac{b_2^2}{a_2^2}\frac{\delta b_2}{b_2}\right\} \tag{23}$$

[1] F. W. Grover: Bul. Bu. Stand. vol. 12, Sci. Paper No. 255, p. 372; 1915.

This is not final; it is necessary to express $\delta\alpha/\alpha$ in terms of quantities more directly measurable. The ratio α is obtained by measuring the current ratio I_2'/I_1' when the coils carrying these currents are made coplanar and produce equal axial components of magnetic field H at their common center.

If H be computed on the same assumption which has been made in deriving the formula for $\mathfrak{F}_m$, which is that the current uniformly fills the winding channels, it is found that

$$H_1=\frac{2\pi n_1 I_1'}{a_1}\left[1+\frac{J_1}{a_1^2}+\frac{G_1}{a_1^4}\right] \tag{24}$$

with a similar expression for H_2, where

$$J_i=\frac{c_i^2}{3}-\frac{b_i^2}{2} \text{ and } G_i=\frac{c_i^4}{5}-b_i^2c_i^2+\tfrac{3}{8}b_i^4 (i=1,2) \tag{24a}$$

When $H_1=H_2$ the ratio α is given (to the fourth order) by

$$\alpha=\tau\left\{1-\frac{1}{a_1^2}\left(J_1-\frac{J_2}{\tau^2}\right)+\frac{1}{a_1^4}\left[J_1^2-G_1+\frac{J_1J_2}{\tau^2}+\frac{G_2-2J_2^2}{\tau^4}\right]\right\} \tag{25}$$

where τ is the measured ratio n_2I_2'/n_1I_1'. The first variation of α is found from this (with the same assumptions as were used in getting (23)).

$$\frac{\delta\alpha}{\alpha}=\frac{\delta\tau}{\tau}-\frac{\delta J_1}{a_1^2}+\frac{\delta J_2}{a_2^2}=$$

$$=\frac{\delta\tau}{\tau}-\frac{1}{3}\left\{2\frac{c_1^2}{a_1^2}\frac{\delta c_1}{c_1}-2\frac{c_2^2}{a_2^2}\frac{\delta c_2}{c_2}-\frac{3b_1^2}{a_1^2}\frac{\delta b_1}{b_1}+\frac{3b_2^2}{a_2^2}\frac{\delta b_2}{b_2}\right\} \tag{26}$$

Introducing this value of $\delta\alpha/\alpha$ into (23) gives finally the fractional error in $\mathfrak{F}_m$

$$\frac{\delta\mathfrak{F}_m}{\mathfrak{F}_m}=\frac{1}{2x_0^2}\left\{\beta\frac{\delta\tau}{\tau}+\left[\frac{(1+\beta)\lambda_2}{3}-\beta\right]\left[\frac{c_1^2}{a_1^2}\frac{\delta c_1}{c_1}-\frac{b_1^2}{a_1^2}\frac{\delta b_1}{b_1}\right]+\left[\frac{(1-\beta)\lambda_2}{3}+\beta\right]\right.$$

$$\left.\left[\frac{c_2^2}{a_2^2}\frac{\delta c_2}{c_2}-\frac{b_2^2}{a_2^2}\frac{\delta b_2}{b_2}\right]\right\} \tag{27}$$

This expression is symmetric with respect to the two coils. This may be seen by noting that β and $\delta\tau/\tau$ are antisymmetric, while x_0^2 and λ_2 are symmetric with respect to the subscripts 1 and 2.

V. CORRECTION FOR CURRENT DISTRIBUTION

In this section are found corrections to $\mathfrak{F}_m$ and also to α (in both cases of fourth order), which are made necessary by the facts that (a) the wires are round, leaving some waste space in the winding channels, and (b) the current density may not be uniformly distributed over the section of each wire, as is generally assumed for steady currents. It is in fact possible that a theoretical examination might lend more probability to the so-called natural distribution in which the current density at any point in the wire is inversely proportional to its distance from the axis of the turn.

It will presently be seen that the correction due to current distribution which must be made in $\mathfrak{F}_m$ is compensated for by the correction which must be made in α when it is determined electrically—the resultant of the two corrections is of higher order than the fourth.

To obtain the correction (a) imagine each coil to consist of n_i turns of square wire all in contact, each having a uniform current distribution. Let $\mathfrak{F}_s$ denote the force between two such turns, one in coil no. 1, the other in no. 2. Since there is no waste space, the force $\mathfrak{F}$, equation (6), is $\Sigma\mathfrak{F}_s$, the summation being extended to all possible pairs of turns, one of which is in no. 1, the other in no. 2.

The force $\mathfrak{F}+\delta\mathfrak{F}^1$ between the actual coils which consist of discrete windings with wire of radius ρ_i will be the corresponding sum

$$\mathfrak{F}+\delta\mathfrak{F}^1=\Sigma\mathfrak{F}_c$$

where $\mathfrak{F}_c$ is the force between the two turns of circular wire which have the same mean plane and same mean radius as the pair of turns of square wire to which $\mathfrak{F}_s$ refers, the current being assumed to be uniform in both cases. If their mean planes are z_1 and z_2 and mean radii are r_1 and r_2, it is found by an expansion similar to (7) that $\mathfrak{F}_c-\mathfrak{F}_s=$

$$\frac{-I_1I_2F(x_1\alpha)}{12x^2}\left\{\left(\frac{c_1^2}{a_1^2}\epsilon_1'-\frac{c_2^2}{a_2^2}\epsilon_2'\right)\beta+\lambda_1\left[(x^2-\beta)\frac{c_1^2}{a_1^2}\epsilon_1'+(x^2+\beta)\frac{c_2^2}{a_2^2}\epsilon_2'\right]\right\} \quad (28)$$

$+$ terms of higher order than the fourth $\left(\text{where } x=\frac{z_1-z_2}{A}=\frac{z}{A}\right)$.

The term retained is in fact of the fourth order since the small quantities ϵ_i' are of the second order, defined by

$$\epsilon_i'=-\frac{a_i^2}{r_i^2}\cdot\frac{b_i}{nc_i}\left(1-\frac{3\rho_i^2n_i}{4b_ic_i}\right) \quad (28')$$

where $2\sqrt{\frac{b_ic_i}{n_i}}$ is the side of the square wire section and $2\,\rho_i$ the diameter of the circular wire. Since $\pi\rho_i^2n_i$ is the total metallic cross section, the fraction $1-\frac{3\rho_i^2n_i}{4b_ic_2}$ is approximately the fraction of cross section which is waste space. For ordinarily close windings, this will be of the order of magnitude (but never less than) $1-\frac{\pi}{4}=0.21$. Hence if the ratio of the diameter of the wire to a side of the cross section of the winding channel is of the same order as b_1/a_1, i.e., of the first order, then $1/n_1$ and ϵ' will be of the second order and the terms above representing $\mathfrak{F}_c-\mathfrak{F}_s$ are of the fourth order, and we may place $a_i^2/r_i^2=1$, $x=x_o$, which gives

$$\epsilon_i'=\frac{3}{4}\frac{\rho_i^2}{c_i^2}-\frac{b_i}{n_ic_i} \quad (i=1,2) \quad (29)$$

and

$$\mathfrak{F}_c=\mathfrak{F}_s-\frac{I_1I_2F(x_o,\,\alpha)}{12x_0^2}\,\beta\left(\frac{c_1^2}{a_1^2}\epsilon_1^1-\frac{c_2^2}{a_2^2}\epsilon_2^1\right) \text{ (since } \lambda_1(x_o,\,\alpha)=0).$$

Summing the n_1n_2 elementary forces between pairs of turns of this type gives

$$\Sigma\mathfrak{F}_c=\mathfrak{F}_m+\delta\mathfrak{F}_m{}^1=\Sigma\mathfrak{F}_s-\frac{n_1n_2I_1I_2F(x_o,\alpha)}{12x_0^2}\,\beta\left(\frac{c_1^2}{a_1^2}\epsilon_1^1-\frac{c_2^2}{a_2^2}\epsilon_2^1\right)$$

Hence, since $\Sigma\mathfrak{F}_s=\mathfrak{F}_m$, we find for this fourth-order correction

$$\frac{\delta\mathfrak{F}_m^1}{\mathfrak{F}_m}=-\frac{\beta}{12x_0^2}\left(\frac{c_1^2}{a_1^2}\epsilon_1^1-\frac{c_2^2}{a_2^2}\epsilon_2^1\right) \tag{30}$$

This takes account of the fact that the coils are wound with wire of circular section having a certain amount of waste space between them. This $\delta\mathfrak{F}_m'$ must be added to the second member of (18) to get the maximum force between the actual coils.

For the correction (b), consider the current density i_1 in a round wire in coil no. 1 to be a function of the distance r_1' of a point in this turn from the axis of the turn. Then expanding to the second order we find

$$i_1=\frac{I_1}{\pi\rho_1^2}\left[1+C_1\left(\frac{r_1^1-r_1}{r_1}\right)+\frac{C_1^1\,\frac{\rho_1^2}{4}-(r^1-r_1^2)}{r_1^2}\right]. \tag{31}$$

For uniform distribution $C_1=C'_1=0$; for the natural distribution $C_1=C_1'=-1$. If F_s' is the force between two circular turns when the current distribution is given by (31) and F_s as defined above, the force between them when their current is uniformly distributed, we find with an expansion similar to (7) by integrating over their sections that $F_s'-F_s$ is a small quantity of the fourth order which is obtained by replacing the ϵ_i' in the second member of (28) by ϵ_i'' where

$$\epsilon_i''=\frac{3}{2}C_i\frac{\rho_i^2}{c_i^2} \tag{32}$$

Hence by a similar argument the correction to the force F_m due to this cause is given by

$$\frac{\delta F_m''}{F_m}=-\frac{\beta}{12x_0^2}\left(\frac{c_1^2}{a_1^2}\epsilon_1''-\frac{c_2^2}{a_2^2}\epsilon_2''\right) \tag{33}$$

The correction $\delta^*\mathfrak{F}_m=\delta\mathfrak{F}_m'+\delta\mathfrak{F}_m''$ due to waste space and to non-uniformity of current in each wire is given by

$$\frac{\delta^*\mathfrak{F}_m}{\mathfrak{F}_m}=-\frac{\beta}{12x_o^2}\left(\frac{c_1^2}{a_1^2}\epsilon_1-\frac{c_2^1}{a_2^2}\epsilon_2\right)\ \textit{where}\ \epsilon_i=\epsilon_i'+\epsilon_i''. \tag{34}$$

The details in computing these two effects on the magnetic field H produced by the coils are so similar to those just sketched that they need not be given. The final result is that (24) is only altered by a change in J_i so that $(24)^a$ has to be replaced by

$$J_i=\frac{c_i^2}{3}\left(1-\frac{\epsilon_i}{2}\right)-\frac{b_i^2}{2} \tag{35}$$

The result of this change is to add a term $\delta^*\alpha/\alpha$ to the second member of (26) where

$$\frac{\delta^*\alpha}{\alpha}=\frac{1}{6}\left(\frac{c_1^2}{a_1^2}\epsilon_1-\frac{c_2^2}{a_2^2}\epsilon_2\right) \tag{36}$$

The contribution of $\delta^*\alpha/\alpha$ to the second member of (23) is

$$\frac{\beta}{12x_o^2}\left[\left(\frac{c_1}{a_1}\right)^2\epsilon_1-\left(\frac{c_2}{a_2}\right)^2\epsilon_2\right],$$

which is exactly canceled by the correction (34) to $\mathfrak{F}_m$ which must be made at the same time. Thus, after taking account of these two distribution effects (to the fourth order), we come back to precisely the earlier equation (27) for the fractional error in the force. But this equation (27) was derived from the expression (18) for the maximum force. Moreover (18) was derived by assuming that the current uniformly filled the winding channels, that is, by neglecting both of these distribution effects. We arrive then at the following conclusion.

VI. CONCLUSION

If the ratio α of the mean radii of the coils of the current balance is determined electrically and this value is used in the computation of the maximum force between the coils using equation (18), the force thus computed will differ from the maximum force between the *actual* coils by an amount which, compared to this force, is a small quantity of order higher than the fourth in the small quantities $\frac{b_1}{a_1}$, $\frac{c_1}{a_1}$, $\frac{b_2}{a_2}$, and $\frac{c_2}{a_2}$, where 2 b_i and 2 c_i are the axial breadth and radial depth respectively of the rectangular cross sections of the winding channels—provided also that the number of turns n_i in each coil is so large that $1/n_i$ is in each case $i=1$ and $i=2$ a small quantity of the order of $(b_i/a_i)^2$ or smaller. These conditions are met in practice. The effects of errors in measurement of cross sectional dimensions may be computed by equation (27). This conclusion rests upon the derivation of equation (18) and also upon the investigation of section V, where account is taken of the facts that the actual coils consist of many turns of round wire with waste space between them, and that the current may not be uniformly distributed over the section of the wires (although the distribution is assumed to be of the same nature in all wires). This section appears, therefore, to give a greater validity to the force equation (18) than was to be expected from the mode of its derivation for it was derived on the assumption that the current uniformly filled the winding channels. The crux of the matter is the electrical determination of the ratio of the radii. If this ratio α were determinable with the desired precision by geometrical measurements of the coils, the conclusion stated above would not be true. In that case, the computation of the force would require a knowledge of the correction term δ^*F_m of (34), which would then be added to the second member of (18).

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